Fourier–Mukai Theory in Commutative Algebra

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Preliminary Oral Examination
June 16, 2021

Section 1

Motivation

Splitting of Vector Bundles on $\mathbb{P}^n$

Let $S = \mathbb{k}[x_0, \ldots, x_n]$ be a polynomial ring.
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- Grothendieck: any vector bundle on $\mathbb{P}^1$ splits as $\oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i)$.
- There are indecomposable bundles of rank $n-1$ on $\mathbb{P}^n$, $n \geq 3$.

Notation:
- $\mathcal{O}_X$ is the structure sheaf on X;
- Ω_X is the cotangent sheaf on X;

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$$\mathcal{O}_{\mathbb{P}^4}^5 \leftarrow \left(\Omega^2_{\mathbb{P}^4}(2)\right)^2 \leftarrow \mathcal{O}_{\mathbb{P}^4}(-1)^5.$$

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- Existence of such bundles on $\mathbb{P}^n$, $n \geq 5$ is unknown.
- Physics: an *instanton bundle* is the cohomology of a *linear* monad

$$\mathcal{O}_{\mathbb{P}^3}(1)^n \leftarrow \mathcal{O}_{\mathbb{P}^3}^{2n+2} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-1)^n.$$

A Splitting Criterion for $\mathbb{P}^n$

Theorem (Horrocks' splitting criterion)

If for all twists $d \in \mathbb{Z}$ and $i \geq 0$, $\mathrm{H}^i(\mathbb{P}^n, E \otimes \mathcal{O}_{\mathbb{P}^n}(d))$ is equal to the cohomology of positive sums of line bundles, then E splits.

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Motivation

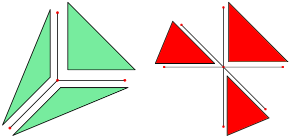
For which toric varieties can we prove a similar splitting criterion?

Section 2

Toric Varieties

Toric Varieties

Let $\sigma \subset N \cong \mathbb{Z}^n$ be a cone
and $\sigma^\vee \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ be its dual cone,
where $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.



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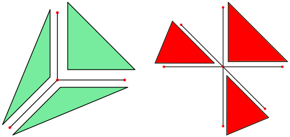

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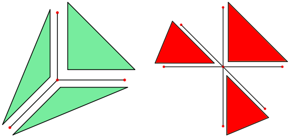

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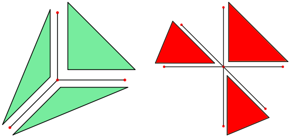

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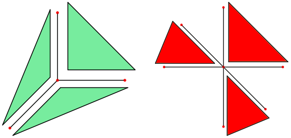

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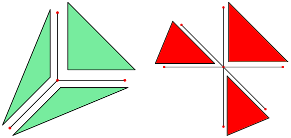

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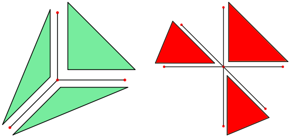

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Definition

A toric variety is *smooth* (resp. *simplicial*) if every cone is generated by a subset of an $\mathbb{Z}$-basis (resp. $\mathbb{R}$-basis) of N (resp. $N \otimes_{\mathbb{Z}} \mathbb{R}$).

- In particular, smooth implies simplicial.

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Caution: saturation on S is with respect to an *irrelevant ideal* B.

Section 3

Derived Categories

Monads We Know & Love!

If we consider longer monads, we can also represent all sheaves.

- Beilinson monads: $B_i = \bigoplus_{j \in \mathbb{Z}} \mathrm{H}^{j-i}(\mathbb{P}^n, E \otimes \mathcal{O}(-j)) \otimes \Omega^j(j)$
- Virtual resolutions: $V_i = \bigoplus_{j \in \mathbb{Z}} \mathrm{H}^{j-i}\left(\mathbb{P}^n, E \otimes \Omega^j(j)\right) \otimes \mathcal{O}(-j)$

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Proposition (Beilinson 1978)

There exit two full strong exceptional collections for $\mathcal{D}^{\mathrm{b}}(\mathbb{P}^n)$:

$$\mathcal{O}_{\mathbb{P}^n},\ \mathcal{O}_{\mathbb{P}^n}(1),\ \ldots,\ \mathcal{O}_{\mathbb{P}^n}(n) \quad \textit{and} \quad \mathcal{O}_{\mathbb{P}^n},\ \Omega_{\mathbb{P}^n}(1),\ \ldots,\ \Omega^n_{\mathbb{P}^n}(n)$$

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Question

What is the machinery for translating between the monads above?

Fourier–Mukai Transforms

Consider the diagram:

$$\begin{array}{ccccc} & & X \times X & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ X & & & & X \end{array}$$

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Let $\mathcal{K}$ be a *resolution of the diagonal* $\Delta = \operatorname{im}(X \to X \times X)$

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Definition (Huybrechts 2006)

The *Fourier–Mukai transform* with kernel $\mathcal{K}$ is the functor

$$\Phi_{\mathcal{K}} : \mathcal{D}^{\mathrm{b}}(X) \to \mathcal{D}^{\mathrm{b}}(X),$$

given by $\quad E \mapsto \pi_{1*}(\pi_2^* E \otimes \mathcal{K}).$

- identity functor on $\mathcal{D}^{\mathrm{b}}(X)$ produces quasi-isomorphisms.

Section 4

Applications

Virtual Resolutions for $X = \mathbb{P}^{\underline{\mathbf{n}}}$

Let $\mathbb{P}^{\underline{\mathbf{n}}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a product of r projective spaces.
Let $M = \bigoplus_d M_d$ be a f.g. $\mathbb{Z}^r$-graded module over the Cox ring S.

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But a **virtual** *Hilbert Syzygy Theorem* for $\mathbb{P}^{\underline{n}}$ still holds:

Theorem (Berkesch, Erman, and Smith 2020)

A virtual resolution of length $\leq \dim \mathbb{P}^n$ *exists for graded modules.*

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"Variations on the theme of Beilinson's resolution of the diagonal."

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We can improve a result of Eisenbud and Goto:

Theorem

$$G_\bullet \cong F_\bullet \quad \text{and} \quad h^{a-j}\left(\widetilde{M} \otimes \Omega^a(a)\right) = \dim \operatorname{Tor}_j(M, k)_a$$

Regularity on X and New Results

Let X be a "nice" toric variety, such as:

- a projective space $\mathbb{P}^n$, or product of projective spaces $\mathbb{P}^{\mathbf{n}}$;
- a Hirzebruch surface $\mathbb{F}_n$;
- the del Pezzo surface $\mathrm{dP6}$.

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$$G_\bullet \cong F_\bullet \quad \text{and} \quad h^{|a|-j}\left(\widetilde{M} \otimes \Omega^a(a)\right) = \dim \operatorname{Tor}_j(M, k)_a$$

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Regularity and Linear Resolutions for $X = \mathbb{P}^n$

Let $S = \mathbb{k}[x_0, \dots, x_n]$ and $\mathfrak{m} = (x_0, \dots, x_n)$ with $\deg x_i = 1$.
Let $M = \bigoplus_d M_d$ be a f.g. $\mathbb{Z}$-graded S-module.

Proposition (Eisenbud and Goto 1984)

The following are equivalent to M being m-regular:

1. *i-th syzygy of M is generated in degrees $\leq m + i$*
2. $H^i_{\mathfrak{m}}(M)_d = 0$ *for* $d \geq m + 1 - i$ *and all* $i = 0, 1, \dots$
3. *the truncation $M_{\geq m}$ admits a linear free resolution.*

Recall that a free resolution $F_\bullet$ of $M_{\geq m}$ is linear if:

- $M_{\geq m}$ is generated in one degree only and
- $F_\bullet$ has only linear elements in its differential matrices

Equivalent to the Betti table being concentrated in one line.

Multigraded Regularity for $X = \mathbb{P}^{\mathbf{n}}$

Let $X = \mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a product of r projective spaces.
Let $S = \mathrm{Cox}(X)$ with B the $\mathbb{Z}^r$-graded irrelevant ideal of X.
Let $M = \bigoplus_d M_d$ be a f.g. $\mathbb{Z}^r$-graded S-module.

Definition (Maclagan and Smith 2004)

An S-module M on a product of projective spaces is **m**-regular if

$$H^i_B(M)_{\mathbf{d}} = 0 \text{ for } \mathbf{d} \in \mathbf{m} + \mathbb{N}^r[1-i] \text{ and all } i = 0, 1, \ldots .$$

Then $\operatorname{reg} M = \{\mathbf{m} \in \mathbb{Z}^r : M \text{ is } \mathbf{m}\text{-regular}\}$.

Notation:

$$\mathbb{N}^r[1-i] = \bigcup \{\mathbb{N}^r \text{ shifted northwest by } 1-i \text{ steps}\}.$$

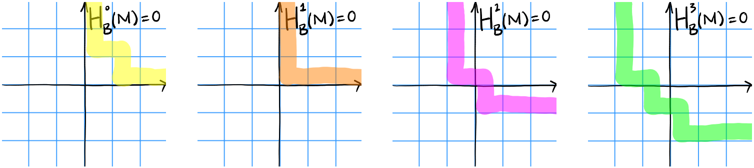
Multigraded Regularity for $X = \mathbb{P}^1 \times \mathbb{P}^2$

Definition (Maclagan and Smith 2004)

$\mathbf{m} \in \operatorname{reg} M \iff H^i_B(M)_{\mathbf{d}} = 0$ for $\mathbf{d} \in \mathbf{m} + \mathbb{N}^2[1-i]$ and all $i \geq 0$.

Notation: $\mathbb{N}^2[1-i] = \bigcup\{\mathbb{N}^2$ shifted northwest by $1-i$ steps$\}$.

Example for Picard rank $= 2$



- Implemented in the M2 package `VirtualResolutions` using Tate resolutions joint with Ayah Almousa, Juliette Bruce, and Mike Loper.